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Computation for Topological Degree and Its Applications

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1. INTRODUCTION

Let Ω be a bounded open set in a Banach space X and $A: \overline{\Omega} \to X$ be a condensing operator, where $\overline{\Omega}$ denotes the closure of Ω in X. In this paper, we first employ the theory of cones to give a new method of computation for topological degree $\deg(I - A, \Omega, \theta) = 0$, where θ denotes the zero element of X. The difference between this note and the well-known references (cf. [**1**, 3–5, 8]) is that we needn't assume A to be a cone mapping, and not even require that there is partial order in X. Then, as applications we study a class of superlinear system of integral equations given by

$$\begin{aligned} \varphi(x) &= \lambda \int_G k_1(x,y) f_1(x, \varphi(y), \psi(y))\, dy \\ \psi(x) &= \lambda \int_G k_2(x,y) f_2(x, \varphi(y), \psi(y))\, dy, \end{aligned} \tag{1}$$

where $\lambda > 0$ is a parameter, G is a bounded closed domain in R^n, k_i is a nonnegative continuous function, and f_i is a continuous function $(i = 1, 2)$. We shall prove that for any positive number λ, (1) has at least a nontrivial solution.

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The following well-known lemmas are very crucial in our arguments, see [**2**] for the proof and further discussion of the topological degree of condensing fields.

LEMMA 1. *If there exists* $u_0 \in X$, $u_0 \neq \theta$, *such that*

$$x - Ax \neq \tau u_0, \qquad \forall x \in \partial\Omega, \tau \geq 0,$$

where $\partial\Omega$ *denotes the boundary of* Ω, *then*

$$\deg(I - A, \Omega, \theta) = 0. \tag{2}$$

LEMMA 2. *If* $\theta \in \Omega$ *and*

$$x \neq \mu Ax, \qquad \forall x \in \partial\Omega, 0 \leq \mu \leq 1,$$

then

$$\deg(I - A, \Omega, \theta) = 1.$$

2. COMPUTATION FOR TOPOLOGICAL DEGREE

A nonempty convex closed set $W \subset X$ is called a weage if it satisfies the following two conditions:

(i) $x \in W$, $\lambda \geq 0$ implies $\lambda x \in W$;

(ii) there exists $y \in W$, such that $-y \notin W$.

There exist Banach spaces $Y_1, Y_2, \ldots, Y_n$, and P_i is a cone in Y_i ($i = 1, 2, \ldots, n$). For convenience, the partial order in each Y_i, which is induced by P_i, is expressed by $\leq$. For $u \in X$, if we let

$$W(u) = \{x \in X | x + u \in W\}, \tag{3}$$

then, we have the following

THEOREM 1. *Let* A: $\overline{\Omega} \to X$ *be a condensing operator which has no fixed point on* $\partial\Omega$. *Suppose that there exist increasing operators* L_i: $P_i \to P_i$, *linear operators* N_i: $W \to P_i$ ($i = 1, 2, \ldots, n$) *and linear operator* T: $W \to W$ *which satisfy the following conditions*:

(i) *there exist* $u_0 \in W \setminus \{\theta\}$ *and natural numbers* m_i, *such that*

$$N_i T^{m_i} u_0 \geq N_i u_0, \qquad N_i u_0 \neq \theta, i = 1, 2, \ldots, n; \tag{4}$$

(ii) *there exists* $u^* \in W$ *such that*

$$A(\partial\Omega) \subset W(u^*); \tag{5}$$

(iii) *for each* i, $1 \leq i \leq n$,

$$N_i Tx = L_i N_i x, \qquad \forall x \in W; \tag{6}$$

(iv) *for any* $x \in \partial\Omega \cap W(u^*)$, *there exists* i_0 *which is dependent on* x $(1 \leq i_0 \leq n)$, *such that*

$$N_{i_0}(Ax + u^*) \geq N_{i_0} T(x + u^*). \tag{7}$$

Then $\deg(I - A, \Omega, \theta) = 0$.

Proof. We need to show that

$$x - Ax \neq \tau u_0, \qquad \forall \tau \geq 0,\ x \in \partial\Omega. \tag{8}$$

If otherwise, there exist $x_0 \in \partial\Omega$ and $\tau_0 \geq 0$, such that $x_0 - Ax_0 = \tau_0 u_0$, hence $\tau_0 > 0$ and

$$x_0 + u^* = Ax_0 + u^* + \tau_0 u_0. \tag{9}$$

By virtue of (5) and (3), $Ax_0 + u^* \in W$, by (9) and $u_0 \in W$ we have $x_0 + u^* \in W$, i.e., $x_0 \in \partial\Omega \cap W(u^*)$. From condition (iv) there exists $1 \leq i_0 \leq n$ such that

$$N_{i_0}(Ax_0 + u^*) \geq N_{i_0} T(x_0 + u^*). \tag{10}$$

Applying N_{i_0} to (9), we see by $Ax_0 + u^* \in W$ and $N_{i_0}\colon W \to P_{i_0}$ that

$$N_{i_0}(x_0 + u^*) = N_{i_0}(Ax_0 + u^*) + \tau_0 N_{i_0} u_0 \geq \tau_0 N_{i_0} u_0. \tag{11}$$

Put $\tau^* = \sup\{\tau | N_{i_0}(x_0 + u^*) \geq \tau N_{i_0} u_0\}$. It is easy to know from $N_{i_0} u_0 \neq \theta$ and (11) that $0 < \tau \leq \tau^* < +\infty$ and

$$N_{i_0}(x_0 + u^*) \geq \tau^* N_{i_0} u_0. \tag{12}$$

By (9), (10), and (6) we find

$$\begin{aligned} N_{i_0}(x_0 + u^*) &= N_{i_0}(Ax_0 + u^*) + \tau_0 N_{i_0} u_0 \\ &\geq N_{i_0} T(x_0 + u^*) + \tau_0 N_{i_0} u_0 \\ &= L_{i_0} N_{i_0}(x_0 + u^*) + \tau_0 N_{i_0} u_0. \end{aligned} \tag{13}$$

Consequently,

$$N_{i_0}(x_0 + u^*) \geq L_{i_0} N_{i_0}(x_0 + u^*).$$

This implies, in view of the fact that $L_i \colon P_{i_0} \to P_{i_0}$ is increasing, that

$$L_{i_0} N_{i_0}(x_0 + u^*) \geq L_{i_0}^2 N_{i_0}(x_0 + u^*). \tag{14}$$

It follows therefore from (13), (14), and (12) that

$$\begin{aligned} N_{i_0}(x_0 + u^*) &\geq L_{i_0}^{m_{i_0}} N_{i_0}(x_0 + u^*) + \tau_0 N_{i_0} u_0 \\ &\geq L_{i_0}^{m_{i_0}}\left(\tau^* N_{i_0} u_0\right) + \tau_0 N_{i_0} u_0 \\ &= L_{i_0}^{m_{i_0}} N_{i_0}(\tau^* u_0) + \tau_0 N_{i_0} u_0. \end{aligned} \tag{15}$$

From (6) we obtain

$$L_{i_0}^{m_{i_0}} N_{i_0} x = N_{i_0} T^{m_{i_0}} x, \qquad \forall x \in W. \tag{16}$$

Thus, (15), (16), and (4) imply

$$\begin{aligned} N_{i_0}(x_0 + u^*) &\geq N_{i_0} T^{m_{i_0}}(\tau^* u_0) + \tau_0 N_{i_0} u_0 \\ &= \tau^* N_{i_0} T^{m_{i_0}} u_0 + \tau_0 N_{i_0} u_0 \\ &\geq \tau^* N_{i_0} u_0 + \tau_0 N_{i_0} u_0 \\ &= (\tau^* + \tau_0) N_{i_0} u_0, \end{aligned}$$

which contradicts the definition of τ^*. Hence (8) is true. Lemma 1 implies that (2) holds.

The following result is an easy consequence of Theorem 1.

COROLLARY 1. *Let $A \colon \overline{\Omega} \to X$ be a condensing operator which has no fixed point on $\partial\Omega$. Suppose that there exist linear operators $T \colon W \to W$ and $N_i \colon W \to P_i$ $(i = 1, 2, \ldots, n)$, such that*

(i) *$N_i u_0 \neq \theta$, for some $u_0 \in W \setminus \{\theta\}$, $i = 1, 2, \ldots, n$;*

(ii) *$A(\partial\Omega) \subset W(u^*)$ for some $u^* \in W$;*

(iii) *for each i, $1 \leq i \leq n$,*

$$N_{i_0} T x = N_i x, \qquad \forall x \in W; \tag{17}$$

(iv) *for any $x \in \partial\Omega \cap W(u^*)$, there exists i_0 which is independent of x $(1 \leq i_0 \leq n)$, such that*

$$N_{i_0} A x \geq N_{i_0} T x. \tag{18}$$

Then (2) *holds*.

Proof. It is clear that the conditions (i), (ii), and (iii) of Theorem 1 are satisfied with $m_i = 1$ and $L_i = I$ $(i = 1, 2, \ldots, n)$. Since N_{i_0} is a linear operator, we find by (17) and (18) that condition (iv) of Theorem 1 holds. The conclusion follows from Theorem 1.

Remark 1. In Theorem 1 and Corollary 1, we do not assume that A is a cone mapping and do not even require that there is partial order in X. Therefore, Theorem 1 gives a computational method of topological degree by applying the theory of cones to studying non-cone mappings, and provides a useful tool for applying the theory of cones to investigating the superlinear Hammerstein system of equations in entire space.

3. POSITIVE EIGENVECTOR OF THE SUPERLINEAR SYSTEM OF INTEGRAL EQUATIONS

In this section we discuss the superlinear Hammerstein system of integral equations (1). If we let

$$A_i(\varphi, \psi)(x) = \int_G k_i(x, y) f_i(y, \varphi(y), \psi(y))\, dy, \qquad i = 1, 2,$$

and $A(\varphi, \psi) = (A_1(\varphi, \psi), A_2(\varphi, \psi))$, then (1) is equivalent to the following eigenvalue problem

$$\bar{\lambda}(\varphi, \psi) = A(\varphi, \psi), \qquad \bar{\lambda} = \frac{1}{\lambda}. \tag{19}$$

Using the results of Section 2, we shall show that any positive real number is an eigenvalue of (19) corresponding to an eigenvector.

Let us list some conditions for convenience:

(H_1) there exist constants $l_i > 0$, such that

$$f_i(x, u, v) \geq -l_i, \qquad \forall x \in G, u, v, \in R^1, i = 1, 2; \tag{20}$$

(H_2) $\lim_{u \to +\infty}(f_1(x, u, v)/u) = +\infty$ uniformly with respect to $x \in G, v \in R^1$; $\lim_{v \to +\infty}(f_2(x, u, v)/v) = +\infty$ uniformly with respect to $x \in G, u \in R^1$;

(H_3) let the operator K_i be defined as

$$(K_i \varphi)(x) = \int_G k_i(x, y) \varphi(y)\, dy, \qquad i = 1, 2,$$

and the spectral radiuses $r(K_i) > 0$ $(i = 1, 2)$. Then by the well-known Krein–Rutman Theorem [**7**], there exist continuous functions $g_i(x) \geq 0$, $g_i(x) \not\equiv 0$ such that

$$\int_G k_i(x, y) g_i(x)\, dx = r(k_i) g_i(y), \qquad \forall y \in G, i = 1, 2; \tag{21}$$

(H_4) there exist constants $\delta_i > 0$, such that

$$g_i(y) \geq \delta_i k_i(\tau, y) \qquad \forall \tau, y \in G, i = 1, 2, \tag{22}$$

where $g_i(y)$ are determined by (21);

(H_5) $\lim_{|u|+|v| \to 0}(f_i(x, u, v)/(|u| + |v|)) = 0$, uniformly with respect to $x \in G$, $i = 1, 2$.

THEOREM 2. *Suppose that conditions* (H_1)–(H_5) *hold. Then*

(i) *for any* $\lambda > 0$, (1) *has at least one nontrivial continuous solution* $(\varphi_\lambda, \psi_\lambda)$;

(ii) *there exists S which is a subset of the solution of* (1), *such that*

$$\lim_{\substack{\lambda \to 0+ \\ (\lambda, \varphi_\lambda, \psi_\lambda) \in S}} \|\varphi_\lambda\| + \|\psi_\lambda\| = +\infty, \tag{23}$$

where $\|\varphi_\lambda\| = \max_{x \in G} |\varphi_\lambda(x)|$.

Proof. From the conditions (H_1) and (H_2), it is easy to show that for $\lambda > 0$, there exist constants $a_i > \lambda^{-1} r^{-1}(K_i)$ and $b_i > 0$ $(i = 1, 2)$, such that

$$f_1(x, u, v) \geq a_1 u - b_1, \qquad \forall x \in G, u, v \in R^1, \tag{24}$$

$$f_2(x, u, v) \geq a_2 v - b_2, \qquad \forall x \in G, u, v \in R^1. \tag{25}$$

Let $C(G) = \{\varphi | \varphi(x)$ is continuous on $G\}$, $X = C(G) \times C(G)$. The norm in X is defined as $\|(\varphi, \psi)\|_X = \|\varphi\| + \|\psi\|$, and obviously X is a Banach space. Evidently, λA: $X \to X$ is completely continuous. In Corollary 1, we take $n = 2$, $Y_1 = Y_2 = R^1$, $P_1 = P_2 = [0, +\infty)$, and $W = W_1 \times W_2$, where

$$W_i = \left\{ \varphi \in C(G) | \varphi(x) \geq 0, \int_G g_i(x) \varphi(x)\, dx \geq r(K_i) \delta_i \|\varphi\| \right\},$$

$$i = 1, 2.$$

It is easy to see that W is a cone in X. Let

$$T(\varphi, \psi) = \left(r^{-1}(K_1)K_1\varphi, r^{-1}(K_2)K_2\psi\right),$$

$$N_1(\varphi, \psi) = \int_G g_1(x)\varphi(x)\,dx, N_2(\varphi, \psi) = \int_G g_2(x)\psi(x)\,dx,$$

and $u_0 = u^* = (\lambda u_1, \lambda u_2)$, where

$$u_i(x) = l_i\int_G k_i(x, y)\,dy, \qquad i = 1, 2,$$

and obviously $u_0 \in W \setminus \{\theta\}$. We first show that the linear operator T maps W into W. In fact, for any $(\varphi, \psi) \in W$, we have $\varphi \in W_1$ and $r^{-1}(K_1)(K_1\varphi)(x) \geq 0$. In addition, by virtue of (21) and (22) we get

$$\begin{aligned}
&\int_G g_1(x)r^{-1}(K_1)(K_1\varphi)(x)\,dx \\
&\quad = r^{-1}(K_1)\int_G g_1(x)\,dx\int_G k_1(x, y)\varphi(y)\,dy \\
&\quad = r^{-1}(K_1)\int_G \varphi(y)\,dy\int_G g_1(x)k_1(x, y)\,dx \\
&\quad = \int_G g_1(y)\varphi(y)\,dy \geq \delta_1\int_G k_1(\tau, y)\varphi(y)\,dy \\
&\quad = \delta_1(K_1\varphi)(\tau), \qquad \forall\tau \in G,
\end{aligned}$$

which implies $r^{-1}(K_1)K_1\varphi \in W_1$. Similarly, $r^{-1}(K_2)K_2\psi \in W_2$ and hence $T(\varphi, \psi) \in W$, i.e., $T(W) \subset W$.

Next, we verify that the operator λA satisfies the conditions (i)–(iv) of Corollary 1.

Evidently, the linear operator N_i maps W into P_i $(i = 1, 2)$. By (21) we obtain

$$\begin{aligned}
N_i u_0 &= \int_G g_i(x)\lambda u_i(x)\,dx \\
&= \lambda l_i\int_G g_i(x)\,dx\int_G k_i(x, y)\,dy \\
&= \lambda l_i\int_G dy\int_G g_i(x)k_i(x, y)\,dx \\
&= \lambda l_i r(K_i)\int_G g_i(y)\,dy > 0, \qquad i = 1, 2,
\end{aligned}$$

and therefore the condition (i) is satisfied.

On the other hand, for any $(\varphi, \psi) \in X$, by (20) we find

$$\begin{aligned}
&\lambda A_1(\varphi, \psi)(x) + \lambda u_1(x) \\
&\quad = \lambda \int_G k_1(x, y) f_1(y, \varphi(y), \psi(y))\, dy + \lambda l_1 \int_G k_1(x, y)\, dy \\
&\quad = \lambda \int_G k_1(x, y) \left[f_1(y, \varphi(y), \psi(y)) + l_1 \right] \geq 0. \qquad (26)
\end{aligned}$$

It follows from (21), (20), and (22) that

$$\begin{aligned}
&\int_G g_1(x) \left[\lambda A_1(\varphi, \psi)(x) + \lambda u_1(x) \right] dx \\
&\quad = \lambda \int_G g_1(x)\, dx \int_G k_1(x, y) \left[f_1(y, \varphi(y), \psi(y)) + l_1 \right] dy \\
&\quad = \lambda \int_G \left[f_1(y, \varphi(y), \psi(y)) + l_1 \right] dy \int_G g_1(x) k_1(x, y)\, dx \\
&\quad = \lambda r(K_1) \int_G g_1(y) \left[f_1(y, \varphi(y), \psi(y)) + l_1 \right] dy \\
&\quad \geq \lambda r(K_1)\, \delta_1 \int_G k_1(\tau, y) \left[f_1(y, \varphi(y), \psi(y)) + l_1 \right] dy \\
&\quad = r(K_1)\, \delta_1 \left[\lambda A_1(\varphi, \psi)(\tau) + \lambda u_1(\tau) \right], \qquad \forall \tau \in G
\end{aligned}$$

which, together with (26), implies

$$\begin{aligned}
&\int_G g_1(x) \left[\lambda A_1(\varphi, \psi)(x) + \lambda u_1(x) \right] dx \\
&\quad \geq r(K_1)\, \delta_1 \| \lambda A_1(\varphi, \psi) + \lambda u_1 \|.
\end{aligned}$$

Thus $\lambda A_1(\varphi, \psi) + \lambda u_1 \in W_1$. And we get $\lambda A_2(\varphi, \psi) + \lambda u_2 \in W_2$ in the same way. Hence

$$\lambda A(\varphi, \psi) + u^* = \left(\lambda A_1(\varphi, \psi) + \lambda u_1, \lambda A_2(\varphi, \psi) + \lambda u_2 \right) \in W$$

and therefore condition (ii) is satisfied.

Moreover, for any $(\varphi, \psi) \in W$, we have by (21)

$$
\begin{aligned}
N_1 T(\varphi, \psi) &= \int_G g_1(x) r^{-1}(K_1)(K_1\varphi)(x)\,dx \\
&= r^{-1}(K_1)\int_G g_1(x)\,dx \int_G k_1(x, y)\varphi(y)\,dy \\
&= r^{-1}(K_1)\int_G \varphi(y)\,dy \int_G g_1(x) k_1(x, y)\,dx \\
&= \int_G g_1(y)\varphi(y)\,dy = N_1(\varphi, \psi).
\end{aligned} \tag{27}
$$

Similarly, $N_2 T(\varphi, \psi) = N_2(\varphi, \psi)$. Then condition (iii) is satisfied.

Finally, we put

$$
\varepsilon_i = a_i \lambda r(K_i) - 1, \qquad i = 1, 2,
$$

$$
R > 2\lambda \max_{1 \le i \le 2} \left\{ \|u_i\| + (\varepsilon_i l_i + b_i) \int_G g_i(y)\,dy \right\}
$$

and let

$$
\Omega_R = \{(\varphi, \psi) \in X \mid \|\varphi\| + \|\psi\| < R\}.
$$

Without loss of generality, we may assume that λA has no fixed point on $\partial\Omega_R$. For any $(\varphi, \psi) \in \partial\Omega \cap W(u^*)$, we have $\|\varphi\| + \|\psi\| = R$, $\varphi + \lambda u_1 \in W_1$, and $\psi + \lambda u_2 \in W_2$. Suppose $\|\varphi\| \ge R/2$, because the proof is similar when $\|\psi\| \ge R/2$. In virtue of (27), (24), and (21) we get

$$
\begin{aligned}
&N_1 \lambda A(\varphi, \psi) - N_1 T(\varphi, \psi) \\
&\quad = N_1 \lambda A(\varphi, \psi) - N_1(\varphi, \psi) \\
&\quad = \lambda \int_G g_1(x)\,dx \int_G k_1(x, y) f_1(y, \varphi(y), \psi(y))\,dy - \int_G g_1(x)\varphi(x)\,dx \\
&\quad \ge \lambda \int_G g_1(x)\,dx \int_G k_1(x, y)[a_1\varphi(y) - b_1]\,dy - \int_G g_1(x)\varphi(x)\,dx \\
&\quad = \lambda r(K_1)\int_G g_1(y)[a_1\varphi(y) - b_1]\,dy - \int_G g_1(x)\varphi(x)\,dx \\
&\quad = (\lambda a_1 r(K_1) - 1)\int_G g_1(x)\varphi(x)\,dx - \lambda b_1 r(K_1)\int_G g_1(y)\,dy \\
&\quad = \varepsilon_1 \int_G g_1(x)[\varphi(x) + \lambda u_1(x)]\,dx - \varepsilon_1 \lambda \int_G g_1(x) u_1(x)\,dx \\
&\qquad - \lambda b_1 r(K_1)\int_G g_1(y)\,dy.
\end{aligned} \tag{28}
$$

Since $\varphi + \lambda u_1 \in W_1$, we obtain

$$\begin{aligned}\int_G g_1(x)\left[\varphi(x) + \lambda u_1(x)\right] dx &\geq \delta_1 r(K_1)\|\varphi + \lambda u_1\| \\ &\geq \delta_1 r(K_1)\left[\|\varphi\| - \lambda\|u\|\right] \\ &\geq \delta_1 r(K_1)\left(\frac{R}{2} - \lambda\|u_1\|\right), \qquad (29)\end{aligned}$$

and hence, by (28) and (29) we have

$$\begin{aligned}&N_1\lambda A(\varphi, \psi) - N_1 T(\varphi, \psi) \\ &\quad \geq \varepsilon_1\delta_1 r(K_1)\left(\frac{R}{2} - \lambda\|u_1\|\right) - r(K_1)\lambda(\varepsilon_1 l_1 + b_1)\int_G g_1(y)\, dy \\ &\quad = \frac{R}{2}\varepsilon_1\delta_1 r(K_1) - \lambda\varepsilon_1\delta_1 r(K_1)\|u_1\| - \lambda r(K_1)(\varepsilon_1 l_1 + b_1)\end{aligned}$$

$\int_G g_1(y)\, dy > 0.$

Consequently, condition (iv) holds. Thus, we have verified that conditions (i)–(iv) of Corollary 1 are satisfied, which implies

$$\deg(I - \lambda A, \Omega_R, \theta) = 0. \qquad (30)$$

Now, let $M = \max\{M_1, M_2\}$, where $M_i = \max_{(x,y)\in G} k_i(x,y)$, $i = 1, 2$. It follows from condition (H_5) that, for given $\alpha_0 = 1/4\lambda M mes G$, there exists $0 < r_\lambda < R$, such that

$$\left|f_i(x, u, v)\right| < (|u| + |v|)\alpha_0, \qquad \forall x \in G, |u| + |v| < r_\lambda. \qquad (31)$$

Taking $\Omega_\lambda = \{(\varphi, \psi) \in X \mid \|\varphi\| + \|\psi\| < r_\lambda\}$, we prove

$$\mu\lambda A(\varphi, \psi) \neq (\varphi, \psi), \qquad \forall\mu \in [0,1], (\varphi, \psi) \in \partial\Omega_\lambda. \qquad (32)$$

In fact, for $\mu \in [0,1], (\varphi, \psi) \in \partial\Omega_\lambda$, by (31) we infer

$$\|\mu\lambda A_i(\varphi, \psi)\| \leq \lambda M\alpha_0(\|\varphi\| + \|\psi\|) mes G < \frac{r_\lambda}{4}, \qquad i = 1, 2,$$

which implies

$$\begin{aligned}\|\mu\lambda A(\varphi, \psi)\|_X &= \|\mu\lambda A_1(\varphi, \psi)\| + \|\mu\lambda A_2(\varphi, \psi)\| \\ &< \frac{r_\lambda}{2} < \|(\varphi, \psi)\|_X,\end{aligned}$$

and hence (32) holds. An application of Lemma 2 shows that

$$\deg(I - \lambda A, \Omega_\lambda, \theta) = 1. \tag{33}$$

It follows from (30), (33), and the additivity property of topological degree that

$$\deg\big(I - \lambda A, \Omega_R \setminus \overline{\Omega}_\lambda, \theta\big) = -1.$$

Thus, by the solution property of topological degree, λA has at least one fixed point in $\Omega_R \setminus \overline{\Omega}_\lambda$. This means that, for any $\lambda > 0$, there exists $(\varphi_\lambda, \psi_\lambda) \in X \setminus \overline{\Omega}_{r_0}$ such that

$$(\varphi_\lambda, \psi_\lambda) = \lambda A(\varphi_\lambda, \psi_\lambda).$$

Therefore, the conclusion (i) of Theorem 2 holds.

Let $r > 0$ be given. Put $m_i = \max_{|u|+|v| \le r} |f_i(x, u, v)|$, $i = 1, 2$, $m = \max\{m_1, m_2\}$, $\Omega_r = \{(\varphi, \psi) | (\varphi, \psi) \in X, \|\varphi\| + \|\psi\| < r\}$, and $\bar{\lambda} = r/2mM \cdot mesG$. For any $\lambda \in (0, \bar{\lambda})$, choosing sufficiently large $R_\lambda > r$, in a way similar to the proof of the conclusion (i), we can show that (1) has one solution $(\varphi_\lambda, \psi_\lambda) \in \Omega_{R_\lambda} \setminus \overline{\Omega}_r$, where $\Omega_{R_\lambda} = \{(\varphi, \psi) \in X | \|\varphi\| + \|\psi\| < R_\lambda\}$. Let

$$S = \Big\{(\lambda, \varphi_\lambda, \psi_\lambda) | \lambda \in (0, \bar{\lambda}), (\lambda, \varphi_\lambda, \psi_\lambda) \text{ satisfies } (1), \|\varphi_\lambda\| + \|\psi_\lambda\| > r\Big\},$$

and we prove that (23) holds. In fact, if (23) is not true, then there exist $(\lambda_n, \varphi_n, \psi_n) \in S$ $(n = 1, 2, \dots)$ such that $\lambda_n \to 0$ $(n \to \infty)$ and $\|\varphi_n\| + \|\psi_n\| < c$, where c is some constant. So $\{(\varphi_n, \psi_n)\}$ has a convergent subsequence, and we can assume $\{(\varphi_n, \psi_n)\}$ converges to $(\varphi^*, \psi^*) \in X$. Since $(\lambda_n, \varphi_n, \psi_n) \in S$, we have

$$(\varphi_n, \psi_n) = \lambda_n A(\varphi_n, \psi_n) \tag{34}$$

and $\|\varphi_n\| + \|\psi_n\| > r$. Taking $n \to \infty$ in (34), we get $(\varphi^*, \psi^*) = \theta$, in contradiction with $\|\varphi^*\| + \|\psi^*\| \ge r > 0$. Thus (23) holds and therefore the conclusion (ii) is true. The proof of the theorem is complete.

Remark 2. Usually, the theory of cone is applied to studying the superlinear system of Hammstein integral equations, and it is necessary to assume that at least one of the following conditions holds:

(1) $f_1(x, u, v) \ge 0$ when $u \ge 0$;

(2) $f_2(x, u, v) \ge 0$ when $v \ge 0$.

But in Theorem 2, we delete the above restriction. Since the operator A is not a cone mapping, it is difficult to obtain Theorem 2 using the fixed point theorem of cone expansion and compression [6] or the fixed point theorem of cone expansion and compression of norm type [3].

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